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Langevin-Like Equation with Colored Noise

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Consider a stochastic differential equation of the form of a Langevin equation, but in which the noise source is not white. If it is nearly white, i.e., its autocorrelation time is short, a systematic approximation method is known. It leads to a Fokker–Planck equation with successive higher order corrections. To obtain the coefficients more explicitly, a secondary expansion may be employed. The validity of the resulting double series approximation is discussed and confronted with the various results given in the literature. In addition, an alternative approximation method is obtained using the technique for eliminating fast variables. It produces the same terms in a different sequence.

KEY WORDS: Colored noise; stochastic differential equations; elimination of fast variables.

1. INTRODUCTION AND CONCLUSIONS

The subject of this study is the stochastic differential equation

$$\dot{x} = \varphi(x) + \psi(x)\,\xi(t) \tag{1}$$

where φ and ψ are given functions of x, and $\xi(t)$ is a given stationary stochastic process with zero mean. If $\xi(t)$ is Gaussian white noise, the equation is equivalent to a Fokker–Planck equation for the probability density $P(x, t)$ of x at time t, that is, an equation of type

$$\dot{P} = \mathscr{L}P \tag{2}$$

with the linear operator $\mathscr{L}$ given by

$$\mathscr{L} = -\frac{\partial}{\partial x}\varphi + \frac{1}{2}\frac{\partial}{\partial x}\psi\frac{\partial}{\partial x}\psi \tag{3}$$

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Now suppose that $\xi(t)$ is not white. [For intrinsic noise one does not expect this to occur without the simultaneous appearance of a memory kernel in (1), but for external noise there is no inconsistency.] For this case a number of authors[1-8] have derived equations similar to (2) and (3) which describe the process $x(t)$ defined by (1) approximately. The ensuing debate has been reviewed in ref. 9. The present paper is an attempt to clarify the situation. The conclusions are here summarized. All results refer to the case that $\xi(t)$ is colored but has a short autocorrelation time τ_c and may therefore be called "off-white."

(i) For off-white $\xi(t)$ the standard treatment of stochastic differential equations leads to an equation (2) with an expansion for $\mathscr{L}$. This I shall call the *primary expansion.* Its first two terms have the form (3) of a Fokker–Planck equation, but the coefficients φ, ψ in it are not quite the same as those in (1).

(ii) To write these new φ, ψ explicitly, a *secondary expansion* may be used. To lowest order one gets the same φ, ψ as those in (1); this is the white noise approximation.

(iii) Both the primary and the secondary expansion involve ascending powers of τ_c. The white noise result is the limit $\tau_c \to 0$. The importance of the higher terms in the primary expansion does not depend on τ_c alone, but also on the P considered. For the stationary solution P^s it is in general inconsistent to ignore them while including higher terms of the secondary expansion.

(iv) An alternative treatment, based on the general method for eliminating fast variables, is possible when $\xi(t)$ is a Markov process. The resulting expansion reproduces the terms of the above double series, but arranged according to their power in τ_c.

(v) The approximate equation obtained in either way may be regarded as defining a Markov process $\tilde{x}(t)$. This process presumably approximates the actual non-Markovian process $x(t)$ in some sense, but the precise character of this approximation is not known.

2. THE INPUT NOISE $\xi(t)$

The process ξ is supposed to have an average zero and an autocorrelation function κ

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\, \xi(t') \rangle = \kappa(t - t')$$

It is also supposed that there is an autocorrelation time τ_c such that $\kappa(\tau)$ is negligible for $|\tau| > \tau_c$. More strongly, it is supposed that one may treat $\xi(t)$ and $\xi(t')$ as statistically independent for $|t - t'| > \tau_c$.

As a measure for the magnitude of ξ I use α, defined by $\alpha^2 = \langle \xi(t)^2 \rangle$. Another relevant quantity is

$$D_0 = \int_0^\infty \kappa(\tau)\, d\tau \sim \alpha^2 \tau_c$$

A special example of such a ξ is the Ornstein–Uhlenbeck process, i.e., the Gaussian Markov process with

$$\kappa(\tau) = \alpha^2 e^{-|\tau|/\tau_c}, \qquad D_0 = \alpha^2 \tau_c \tag{4}$$

This becomes Gaussian white noise in the limit

$$\tau_c \to 0, \qquad \alpha^2 \to \infty, \qquad D_0 = \text{const} \tag{5}$$

3. THE MARKOV APPROXIMATION

Equation (1) defines uniquely a stochastic process $x(t)$ for $t \geqslant t_0$ provided one adds an initial condition; we take $x(0) = x_0$ with a fixed, non-random x_0. To specify a stochastic process, one needs the entire hierarchy of distribution functions[10,11]

$$P_1(x_1, t_1), \quad P_2(x_1, t_1; x_2, t_2), \quad P_3(x_1, t_1; x_2, t_2; x_3, t_3), \ldots \tag{6}$$

Our initial condition is expressed by

$$P_1(x_1, t_0) = \delta(x_1 - x_0) \tag{7}$$

and the aim is to find an appropriate equation of type (2) for the function $P_1(x, t)$.

If $x(t)$ were a Markov process, the entire hierarchy (6) would be known once one knows P_1 and P_2. In lieu of P_2 one may also take the transition probability

$$P(x_2, t_2 \mid x_1, t_1) = \frac{P_2(x_1, t_1; x_2, t_2)}{P_1(x_1, t_1)} \qquad (t_0 < t_1 \leqslant t_2) \tag{8}$$

Moreover, for a Markov process one would have $P_1(x_1, t_1) = P(x_1, t_1 \mid x_0, t_0)$. Hence an equation (2) for P_1 would be at the same time an equation for $P(x_2, t_2 \mid x_1, t_1)$—which is the "master equation."[11] Hence, for a Markov process an equation for P_1 is tantamount to a full specification of the process.

However, $x(t)$ is *not* Markovian, since $\xi(t)$ is not white. The equation for P_1 does not carry over to $P(x_2, t_2 \mid x_1, t_1)$. The reason is that t_1 is not on the same footing as t_0, since only at t_0 is it true that the value of x is uncorrelated with the value of ξ. Moreover, the transition probability does not suffice to construct the hierarchy.[12,13] So why bother about an equation for P_1?

It seems to me that commonly the motivation is the following.[14] Suppose one has found for P_1 an equation of type (2). Suppose also that, with some approximation, $\mathscr{L}$ turns out to be independent of x_0 and t_0. Then this equation has the same form as a master equation. Ignoring the special role of t_0, one proceeds to interpret it as if it were a master equation, thereby defining a Markov process $\tilde{x}(t)$. The tacit assumption is that this $\tilde{x}(t)$ is an approximation to the actual process $x(t)$ determined by (1). It is not the purpose of this paper to investigate this assumption, but merely to review the approximation methods used for obtaining an equation (2) for $P_1(x, t)$.

The *first-passage time* of a Markov process is important in the theory of chemical reactions, etc. For non-Markovian processes it is not clear whether there exists a single quantity with a similar role, nor that it is approximately equal to the first-passage time of $\tilde{x}(t)$. This problem and the related problem of escape over a potential barrier are not considered here.

4. REDUCTION TO A SIMPLER EQUATION

Equation (1) may be simplified by transforming x to a new variable u,

$$u = \int \frac{dx}{\psi(x)}, \qquad \frac{\varphi(x)}{\psi(x)} = f(u)$$

The new form of the equation is

$$\dot{u} = f(u) + \xi(t) \tag{9}$$

Throughout I shall work with (9) rather than (1) to simplify the algebra without losing generality. The only restriction is that $\psi(x)$ must nowhere vanish, as is the case in most applications. If it does have a zero, serious difficulties arise, which require special treatment.[15]

Incidentally, Eq. (9) is often called the "additive" case to distinguish it from the "multiplicative" case (1). This distinction is spurious since they can be transformed into each other. *The term "additive" makes sense only when $\varphi(x)$ is linear*; for then the case $\psi = \text{const}$ has a special significance, namely that the effects on $x(t)$ of the separate fluctuations in $\xi(t)$ add up. If there are more variables, our simplifying transformation is not possible in general, which causes complications.[3]

5. SOLUTION BY THE CUMULANT EXPANSION METHOD

Equations (1) and (9) are stochastic differential equations.[2] The construction of an approximate equation (2) was achieved by Stratonovich.(16) An explicit form is given in ref. 17. I shall refer to this method as *the cumulant expansion of stochastic differential equations.*(3,18,19) It produces an operator $\mathscr{L}$ in the form of an expansion. For the simplified version (9) the first two terms turn out to be

$$\frac{\partial P(u, t)}{\partial t} = -\frac{\partial}{\partial u} f(u) P + \frac{\partial^2}{\partial u^2} D(u) P \tag{10}$$

where the coefficient $D(u)$ is given by

$$D(u) = \int_0^\infty d\tau\, \kappa(\tau) \left(\frac{du^{-\tau}}{du}\right)^{-1} \tag{11}$$

This is the answer to the problem of finding an approximate solution of (1). The last factor in (11) has the following meaning. On solving the deterministic equation

$$\dot{u} = f(u) \tag{12}$$

one obtains for each fixed t a mapping of the initial u onto the value u^t at t. The derivative of this map taken at $t = -\tau$ is $du^{-\tau}/du$.

Actually the calculation leads to (11) with an upper limit of integration $t - t_0$. That makes (10) explicitly dependent on the initial time t_0 and hence unfit to be interpreted as master equation for a Markov process $\tilde{x}(t)$. However, the integrand of (11) is practically zero for $\tau > \tau_c$ and the error made by extending the limit to ∞ is negligible provided that $t - t_0 > \tau_c$. This is where the as yet uncontrolled Markov approximation is made.

The solution of (12) is given by

$$t = \int_u^{u^t} \frac{dv}{f(v)}\, dv \tag{13}$$

If u varies, and u^t varies with it while t is constant,

$$0 = \frac{du^t}{f(u^t)} - \frac{du}{f(u)}$$

[2] This name is often confined to the case that $\xi(t)$ is Gaussian white noise, but I use it for any differential equation whose coefficients are random with given stochastic properties.

Hence (11) may be written alternatively as

$$D(u) = \int_0^\infty \kappa(\tau)\, d\tau \, \frac{f(u)}{f(u^{-\tau})} \tag{14}$$

6. THE SECONDARY EXPANSION

In order to find $D(u)$ explicitly, it is necessary to solve (12), that is, to invert the integral (13)—at least for times up to τ_c. This can always be done by expanding in powers of t. Thus, *in addition to the primary expansion, whose first two terms are (10), we now make a secondary, optional expansion* to facilitate the calculation of $D(u)$.

Solve (12) to second order in t,

$$u^t = u + tf(u) + \tfrac{1}{2}t^2 f'(u)\, f(u) + O(t^3)$$

Hence, with fixed t,

$$du^t/du = 1 + tf' + \tfrac{1}{2}t^2(f'^2 + ff'') + O(t^3)$$

Set $t = -\tau$ and take the reciprocal,

$$(du^{-\tau}/du)^{-1} = 1 + \tau f' + \tfrac{1}{2}\tau^2(f'^2 - ff'') + O(\tau^3)$$

Substitute in (11),

$$D(u) = D_0 + f' \int_0^\infty \tau\kappa(\tau)\, d\tau + \tfrac{1}{2}(f'^2 - ff'') \int_0^\infty \tau^2 \kappa(\tau)\, d\tau + O(\tau_c^3) \tag{15}$$

These are the first three terms of the expansion of $D(u)$ in powers of τ_c.

The higher powers become rapidly more complicated. Much of the literature is concerned with finding a general expression for them (Appendix C). Yet this question seems to me of secondary importance inasmuch as it concerns only the secondary expansion of the function $D(u)$; its exact form is known anyway [see (11)], while the higher terms in the primary expansion are still unknown.

To be honest, however, one must confess that the secondary expansion is not just a substitute for the exact expression. It may happen that the integral in (11) diverges even though $\kappa(\tau)$ decreases exponentially. Yet in that case the separate terms in (15) are still convergent integrals; the primary expansion is strictly speaking meaningless, but after the secondary expansion has been carried out the result has at least a formal meaning. It is hoped that this formal result still produces a sensible approximation.

7. HIGHER TERMS IN THE PRIMARY EXPANSION

The primary expansion is an expansion in $(\alpha\tau_c)$.[17-19] The Fokker–Planck approximation (10) comprises the first and second terms. The general nth term is of order

$$\alpha(\alpha\tau_c)^{n-1} \sim D_0^{n/2}\tau_c^{n/2-1} \qquad (n=2, 3,...)$$

The higher terms involve higher derivatives as well, in general up to $(\partial/\partial u)^n$. This is why (10) with (11) has been called "the best Fokker–Planck approximation"—the lesser ones being (10) with some approximate expression for $D(u)$.

In each term the coefficients may be subjected to a secondary expansion in powers of τ, as in Section 6. The result is a triple series of terms proportional to

$$D_0^{n/2}\tau_c^{n/2-1+m}\left(\frac{\partial}{\partial u}\right)^{n-k} P \qquad (n\geqslant 2; m\geqslant 0; \quad 0\leqslant k\leqslant n-1) \tag{16}$$

This is the general case. If ξ is even, the only nonzero terms are those with even powers of α, that is, with even n. If ξ is Gaussian, the terms with $k=0$, $m=0$ also vanish; see Appendix A.

A first conclusion is that in the white noise limit (5) one is left with the Fokker–Planck equation (10) with $D(u)=D_0$. In terms of the original variable x it has the form (3), which is the Stratonovich result.

Incidentally, the fact that this equation appears strictly as a limiting case explains why the Fokker–Planck equation has a mathematically exact meaning by itself as the master equation of a well-defined Markov process $\tilde{x}(t)$. It is true that Itô did not need this limit since he assigned a meaning to (1) by decree. That also led to an equation (2) for a Markov process, although with $\mathscr{L}$ slightly different from (3). As his decree is not what one gets for $\tau_c\to 0$, but applies exclusively to $\tau_c=0$, that is, to exactly white $\xi(t)$, it has no meaning in physics.

A second conclusion from (16) is that the magnitude of the successive terms is not determined by the parameters D_0 and τ_c alone, but also by the function P under consideration. This is the subject of the next section.

8. ESTIMATE OF THE TERMS IN THE PRIMARY EXPANSION

Whoever wants to improve on the white noise approximation should not just improve the expression for the coefficient $D(u)$ in the Fokker–Planck equation, but also estimate the higher terms of the primary expansion, beyond Fokker–Planck. The expansion parameter $\alpha\tau_c \sim (D_0\tau_c)^{1/2}$ may be called "the jump size," i.e., the amount by which u is perturbed by ξ during one autocorrelation time τ_c of ξ. Moreover, each successive term contains an additional $\partial/\partial u$, which represents a factor $1/\Delta$, where Δ is the distance over which P varies appreciably. Thus, *the principal expansion is based on the smallness of the jumps compared with the variation length of the P considered*:

$$(D_0 \tau_c)^{1/2} \ll \Delta \tag{17}$$

This condition is already apparent in Planck's original derivation.[20] We consider three types of P.

First, when P is a delta function or some other sharp peak, the condition (17) is violated. Hence, one cannot really solve (1) with initial condition (7) using the Fokker–Planck approximation. Yet, after an initial transient time Δ^2/ψ, P will have broadened sufficiently for (17) to hold. That is good enough for the purpose of computing averages of functions that vary little over a distance Δ. A similar situation is well known: the ordinary diffusion equation is customarily solved with initial delta function, although obviously the equation is not valid on a scale smaller than the mean free path. Yet the result is good enough for any observer who cannot distinguish such small-scale phenomena.

Second, suppose one wants to utilize (10), (11) to obtain the stationary distribution P^s. Let (12) have a single attractor, say $u = 0$:

$$f(u) = f'(0)\, u + O(u^2) = -u/\tau_m + O(u^2) \tag{18}$$

The stationary solution produced by (10) is roughly

$$P^s(u) \approx \text{const} \cdot \exp(-u^2/2D_0 \tau_m) \tag{19}$$

Hence $\Delta \sim (D_0 \tau_m)^{1/2}$ and the successive terms of the primary expansion are of order

$$D_0^{n/2} \tau_c^{n/2-1} \Delta^{-n} \sim \tau_m (\tau_c/\tau_m)^{n/2-1}$$

The condition for the validity of the Fokker–Planck approximation is therefore

$$\tau_c \ll \tau_m \tag{20}$$

Note that $\tau_m \sim u/f(u) \sim 1/f'(u)$ is an estimate of the time scale of the deterministic motion. The condition (20) states that the autocorrelation time of $\xi(t)$ must be small compared to it.

Of course other attractors are possible, such as a critical point: $f(u) = -\gamma u^3$. They can be discussed in a similar way.

Third, suppose f has the form (18) but u starts at some u_0 far away. In order that the entire approach toward the ultimate P^s can be described by the Fokker–Planck equation, the condition (20) is certainly necessary. It will also be sufficient (at least after the initial transient) if one knows that the width Δ of $P(u, t)$ is at all times of the same order as the width of $P^s(u)$. This will be the case if $f(u)$ is attractive in the relevant interval of u; more precisely, $f(u)/u$ should be negative. If, on the other hand, $f(u_1) = 0$ at some u_1 between $u = u_0$ and $u = 0$, then there is an unstable equilibrium point of the deterministic equation (12). Such points require special treatment even in the case of white noise; for literature see ref. 21.

Conclusion. There is no universal criterion for the validity of the Fokker–Planck approximation, but (20) is necessary and sufficient for computing how an initial value approaches the equilibrium when there are no unstable or critical points.

9. ESTIMATE OF THE TERMS IN THE SECONDARY EXPANSION

The function $f(u)$ has a variation length $\delta \sim |f/f'|$. The time τ_m needed for u to change by this amount is $\delta/|f| \sim |f'|^{-1}$. Thus, the time scale τ_m of the macroscopic motion is the time in which it becomes apparent that the motion is not uniform. It appears from (15) that *in the secondary expansion the mth term is of order* $(\tau_c/\tau_m)^m$ $(m = 0, 1, 2,...)$. This expansion provides a good approximation if $\tau_c/\tau_m \ll 1$. It was found in (20) that this is also the condition for applying the Fokker–Planck approximation to the stationary solution near an attractor. Hence, *for finding the equilibrium, the so-called best Fokker–Planck equation is no better than the white noise approximation.*

This is true in general, but for special forms of $\xi(t)$ the situation may be somewhat more favorable. In fact, if $\xi(t)$ is Gaussian, the terms with $n = 3$ in (16) vanish and also the term with $n = 4$, $m = 0$ (Appendix A). Hence the corrections to (10) are of order τ_c^2 and it is permitted to include in $D(u)$ the first correction, which is of order τ_c.

Is P^s for nonwhite noise broader or narrower than for white noise? This question makes sense only if one takes for $\xi(t)$ a specific form in which τ_c enters as a parameter. I take Ornstein–Uhlenbeck specified by (4) and vary τ_c as in (5). It is then justified to use (10) with

$$D(u) = D_0[1 + \tau_c f'(u)] = D_0(1 - \tau_c/\tau_m)$$

It follows that P^s is narrower than for white noise to first order in τ_c. To order τ_c^2 one must include the term with $n = 4$ and therefore a fourth derivative of P. As a consequence, P^s is no longer Gaussian and a comparison of widths is moot.

10. ALTERNATIVE TREATMENT BY ELIMINATING THE FAST VARIABLE

So far I have utilized the results of the cumulant expansion of stochastic differential equations. It is also possible to treat ξ as a fast variable and utilize the known method for eliminating fast variables.[22,23] This method, however, works only if $\xi(t)$ is itself a Markov process obeying some master equation. I take for $\xi(t)$ the Ornstein–Uhlenbeck process (4). On eliminating ξ, we shall find an equation of the form (2), in which $\mathscr{L}$ is obtained as a series in powers of τ_c.

To cast the problem in a suitable form, write the equation for the joint probability $F(u, \xi, t)$ of u and ξ,

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial u}\left[f(u) + \xi\right] F + \frac{1}{\tau_c}\left(\frac{\partial}{\partial \xi}\xi + \frac{D}{\tau_c}\frac{\partial^2}{\partial \xi^2}\right) F \tag{21}$$

For small τ_c the fast variable ξ may be eliminated so as to obtain an equation for

$$\int F(u, \xi, t)\, d\xi = P(u, t)$$

However, the result in ref. 23 does not carry over straightaway because there the limit $\tau_c \to 0$ was taken with constant α, whereas we are now concerned with the limit (5). In this limit $\langle \xi^2 \rangle^s \sim D_0/\tau_c$ is large. Hence, *we have to rescale* ξ by setting $\xi = \eta (D_0/\tau_c)^{1/2}$ in order to exhibit the powers of τ_c:

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial u} f(u)\, F - \left(\frac{D_0}{\tau_c}\right)^{1/2} \eta \frac{\partial F}{\partial u} + \frac{1}{\tau_c}\left(\frac{\partial}{\partial \eta}\eta + \frac{\partial^2}{\partial \eta^2}\right) F \tag{22}$$

The calculation is now straightforward; see Appendix B. The result is

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial u} f(u)\, P + D_0 \frac{\partial^2 P}{\partial u^2} + \tau_c D_0 \frac{\partial^2}{\partial u^2} f'(u)\, P \tag{23}$$

One recognizes the white noise approximation and the order-τ_c correction to $D(u)$ according to the secondary expansion (15). To this order of τ_c no contribution from the primary expansion appears because we have chosen $\xi(t)$ to be Gaussian.

APPENDIX A. THE CUMULANT EXPANSION

The higher order terms of the primary expansion can be found by applying the recipe given in refs. 18 and 19; see also ref. 3. First construct the Liouville equation associated with (9),

$$\frac{\partial \rho(u, t)}{\partial t} = -\frac{\partial}{\partial u} f(u)\, \rho - \xi(t) \frac{\partial \rho}{\partial u}$$

It is again a stochastic differential equation, but linear in the unknown.[3] The linear operators A_0 and A_1 in ref. 18 are in the present case

$$A_0 = -\nabla f, \qquad A_1(t) = -\xi(t)\nabla$$

where ∇ stands for $\partial/\partial u$ acting on everything to the right of it. The average $\langle \rho(u, t)\rangle$ is identical with the probability density $P(u, t)$ of u.[24]

The cumulant expansion of linear stochastic equations can now be applied and gives to second order in $\xi(t)$

$$\mathscr{L} = A_0 + e^{tA_0}\int_0^\infty \langle V(t)\, V(t-\tau)\rangle\, d\tau \cdot e^{-tA_0} \tag{A1}$$

The operator e^{tA_0} is determined by its action on any test function $g(u)$:

$$e^{tA_0}g(u) = e^{-t\nabla f}g(u) = \frac{du^{-t}}{du}\, g(u^{-t}) = \frac{f(u^{-t})}{f(u)}\, g(u^{-t})$$

$V(t)$ is the time-dependent stochastic operator defined by

$$V(t) = \alpha e^{-tA_0}A_1(t)\, e^{tA_0} = -\xi(t)\, e^{t\nabla f}\nabla e^{-t\nabla f}$$

Acting on any $g(u)$, it gives

$$V(t)\, g(u) = -\xi(t)\nabla \frac{du}{du^t}\, g(u) = -\xi(t)\nabla \frac{f(u)}{f(u^t)}\, g(u)$$

By applying these identities repeatedly, one obtains

$$\begin{aligned} &e^{tA_0}V(t)\, V(t-\tau)\, e^{-tA_0}g(u) \\ &\quad = \xi(t)\,\xi(t-\tau)\,\nabla e^{-\tau\nabla f}\nabla e^{\tau\nabla f}g(u) \\ &\quad = \xi(t)\,\xi(t-\tau)\,\nabla e^{-\tau\nabla f}\nabla \frac{f(u^\tau)}{f(u)}\, g(u^\tau) \\ &\quad = \xi(t)\,\xi(t-\tau)\,\nabla \frac{du^{-\tau}}{du}\frac{d}{du^{-\tau}}\frac{f(u)}{f(u^{-\tau})}\, g(u) \\ &\quad = \xi(t)\,\xi(t-\tau)\,\nabla^2 \frac{f(u)}{f(u^{-\tau})}\, g(u) \end{aligned} \tag{A2}$$

Substitution in (A1) yields (10), (11).

[3] The name "stochastic Liouville equation" has been used for this equation but also for the entirely different type of equation (21).

The term in $\mathscr{L}$ of order ξ^4 is given by the ordered cumulant (indicated by triple brackets[18,19])

$$\int e^{tA_0} \langle\!\langle\!\langle V(t)\, V(t_1)\, V(t_2)\, V(t_3) \rangle\!\rangle\!\rangle\, e^{-tA_0}\, dt_1\, dt_2\, dt_3 \tag{A3}$$

The integral extends over $t > t_1 > t_2 > t_3$ and the ordered cumulant is (in abbreviated notation)

$$\begin{aligned} \langle\!\langle\!\langle V V_1 V_2 V_3 \rangle\!\rangle\!\rangle = \langle V V_1 V_2 V_3 \rangle - \langle V V_1 \rangle \langle V_2 V_3 \rangle \\ - \langle V V_2 \rangle \langle V_1 V_3 \rangle - \langle V V_3 \rangle \langle V_1 V_2 \rangle \end{aligned}$$

We set $t_i = t - \tau_i$ and use similar tricks as in (A2). The integrand of (A3) becomes

$$\begin{aligned} &(\langle \xi \xi_1 \xi_2 \xi_3 \rangle - \langle \xi \xi_1 \rangle \langle \xi_2 \xi_3 \rangle)\, \nabla^2 \frac{d}{du^{-\tau_1}} \frac{d}{du^{-\tau_2}} \frac{du}{du^{-\tau_3}} \\ &- \langle \xi \xi_2 \rangle \langle \xi_1 \xi_3 \rangle\, \nabla^2 \frac{d}{du^{-\tau_2}} \frac{d}{du^{-\tau_1}} \frac{du}{du^{-\tau_3}} \\ &- \langle \xi \xi_3 \rangle \langle \xi_1 \xi_2 \rangle\, \nabla^2 \frac{d}{du^{-\tau_3}} \frac{d}{du^{-\tau_1}} \frac{du}{du^{-\tau_2}} \end{aligned} \tag{A4}$$

On expressing all derivatives in terms of ∇, one obtains a plethora of terms, which together constitute the next order in the primary expansion beyond the Fokker–Planck approximation. Rather than writing them all down, I select a few simple cases.

First consider the secondary expansion of (A4). To zeroth order in τ_c one has $u^{-\tau_1} = u^{-\tau_2} = u^{-\tau_3} = u$, so that one obtains

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial u} fP + \frac{\partial^2}{\partial u^2} D(u)\, P + D^* \frac{\partial^4 P}{\partial u^4} \tag{A5}$$

$$D^* = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3\, \langle\!\langle\!\langle \xi(t)\, \xi(t_1)\, \xi(t_2)\, \xi(t_3) \rangle\!\rangle\!\rangle \tag{A6}$$

The order of D^* is $\alpha^4 \tau_c^3 \sim D_0^2 \tau_c$, so that the correction term is of the form (16) with $n = 4$, $m = 0$, $k = 0$, as expected. It is readily seen that all higher order terms of the primary expansion, each taken at the lowest order of its secondary expansion, are constructed in the same way from the ordered cumulants of $\xi(t)$.

Second, if $\xi(t)$ happens to be Gaussian, all its cumulants beyond the second vanish and $D^* = 0$. I make the stronger assumption that $\xi(t)$ is an

Ornstein–Uhlenbeck process, and compute the next order of (A4) in the secondary expansion. One has

$$\frac{d}{du^{-\tau_1}}=\frac{f(u)}{f(u^{-\tau_1})}\nabla=\frac{f(u)}{f(u)-\tau_1 f(u)\,f'(u)}\nabla=(1+\tau_1 f')\nabla+O(\tau_c^2)$$

Thus, (A4) is, to first order in the τ's,

$$\begin{aligned}&\langle\xi\xi_2\rangle\langle\xi_1\xi_3\rangle\,\nabla^2(\tau_2-\tau_1)(\nabla f'\nabla-f'\nabla^2)\\&\quad+\langle\xi\xi_3\rangle\langle\xi_1\xi_2\rangle\,\nabla^2[(\tau_1-\tau_3)\,f'\nabla^2+(\tau_2-\tau_1)\,\nabla f'\nabla+(\tau_3-\tau_2)\,\nabla^2 f')]\end{aligned}$$

In both lines the factors involving ξ are $\alpha^4\exp[(\tau_1-\tau_2-\tau_3)/\tau_c]$. Performing the integration over $\tau_3>\tau_2>\tau_1>0$, one obtains

$$\alpha^4\tau_c^4\,\nabla^2(\tfrac{1}{4}f''\nabla-\tfrac{3}{4}f'\nabla^2+\tfrac{1}{4}\nabla f'\nabla+\tfrac{1}{2}\nabla^2 f')$$

That gives to order τ_c^2

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial u}fP+\frac{\partial^2}{\partial u^2}D(u)\,P+\frac{1}{2}D_0^2\tau_c^2\frac{\partial^2}{\partial u^2}\left(3f''\frac{\partial P}{\partial u}+f'''P\right)\qquad\text{(A7)}$$

The new term is the one in (16) with $n=4$, $m=1$, $k=1, 2$.

APPENDIX B. THE ELIMINATION OF THE FAST VARIABLE ξ

Decompose (22) according to the powers of τ_c,

$$\dot{F}=(\tau_c^{-1}\mathscr{L}_0+\tau_c^{-1/2}\mathscr{L}_1+\mathscr{L}_2)\,F\qquad\text{(B1)}$$

The largest term is the one with $\mathscr{L}_0$ and we therefore select the projection operator $\mathscr{P}=1-\mathscr{Q}$ defined by

$$\mathscr{P}F(u,\eta)=m(\eta)\int F(u,\eta')\,d\eta',\qquad m(\eta)=(2\pi)^{-1/2}\,e^{-\eta^2/2}$$

Then

$$\mathscr{P}\mathscr{L}_0=\mathscr{L}_0\mathscr{P}=0,\qquad\mathscr{P}\mathscr{L}_1\mathscr{P}=0,\qquad\mathscr{P}\mathscr{L}_2=\mathscr{L}_2\mathscr{P}$$

Equation (B1) may be split into two coupled equations for $\mathscr{P}F$ and $\mathscr{Q}F$:

$$\begin{aligned}\partial_t\mathscr{P}F&=\mathscr{L}_2\mathscr{P}F+\tau_c^{-1/2}\mathscr{P}\mathscr{L}_1\mathscr{Q}F\\\partial_t\mathscr{Q}F&=\mathscr{L}_2\mathscr{Q}F+\tau_c^{-1/2}\mathscr{Q}\mathscr{L}_1\mathscr{P}F+\tau_c^{-1/2}\mathscr{Q}\mathscr{L}_1\mathscr{Q}F+\tau_c^{-1}\mathscr{Q}\mathscr{L}_0\mathscr{Q}F\end{aligned}$$

The first one can be written

$$\partial_t P(u,t) = \mathscr{L}_2 P + \tau_c^{-1/2} \int d\eta\, \mathscr{L}_1 w \tag{B2}$$

while $w = \mathscr{Q}F$ is the solution of the second one,

$$\partial_t w(u,\eta,t) = \mathscr{L}_2 w + \tau_c^{-1/2} \mathscr{L}_1 m(\eta)\, P + \tau_c^{-1/2} \mathscr{Q}\mathscr{L}_1 w + \tau_c^{-1} \mathscr{Q}\mathscr{L}_0 w$$

Expand $w = \tau_c^{1/2} w_1 + \tau_c w_2 + \tau_c^{3/2} w_3$, so that to order $\tau_c^{1/2}$

$$\begin{aligned}\tau_c^{1/2}\partial_t w_1 = {}& \tau_c^{1/2}\mathscr{L}_2 w_1 + \tau_c^{-1/2}\mathscr{L}_1 mP + \mathscr{Q}\mathscr{L}_1 w_1 + \tau_c^{1/2}\mathscr{Q}\mathscr{L}_1 w_2 \\ & + \tau_c^{-1/2}\mathscr{Q}\mathscr{L}_0 w_1 + \mathscr{Q}\mathscr{L}_0 w_2 + \tau_c^{1/2}\mathscr{Q}\mathscr{L}_0 w_3\end{aligned}$$

The terms of order $\tau_c^{-1/2}$ give

$$w_1 = -\mathscr{L}_0^{-1}\mathscr{L}_1 mP = \mathscr{L}_0^{-1}\, D_0^{1/2}\eta m \frac{\partial P}{\partial u} = -\eta m\, D_0^{1/2} \frac{\partial P}{\partial u}$$

On inserting this into (B2), one finds the first two terms of (23).

Continuing in this way, we obtain

$$\begin{aligned} w_2 = -\mathscr{L}_0^{-1}\mathscr{Q}\mathscr{L}_1 w_1 &= -\mathscr{L}_0^{-1}\, D_0 \frac{\partial}{\partial u}(\eta^2-1)\, m \frac{\partial P}{\partial u} \\ &= \frac{1}{2} D_0(\eta^2-1)\, m \frac{\partial^2 P}{\partial u^2}\end{aligned}$$

It does not contribute to (B2). Furthermore,

$$\begin{aligned}\mathscr{L}_0 w_3 &= -\mathscr{Q}\mathscr{L}_1 w_2 - \mathscr{L}_2 w_1 + \partial_t w_1 \\ &= \frac{1}{2} D_0^{3/2}(\eta^3-\eta)\, m \frac{\partial^3 P}{\partial u^3} - D_0^{1/2}\eta m \frac{\partial}{\partial u} f \frac{\partial P}{\partial u} - D_0^{1/2}\eta m \frac{\partial}{\partial t}\frac{\partial P}{\partial u}\end{aligned} \tag{B3}$$

There is no need to solve this for w_3, because all that is required in (B2) is the integral

$$\int d\eta\, \mathscr{L}_1 w_3 = -D_0^{1/2} \frac{\partial}{\partial u} \int \eta\, d\eta\, \mathscr{L}_0^{-1} R \tag{B4}$$

where R is the right-hand member of (B3). The integral can be evaluated by applying the adjoint of $\mathscr{L}_0^{-1}$ to the factor η, which gives $-\eta$, so that one gets for (B4)

$$D_0^{1/2} \frac{\partial}{\partial u} \int \eta\, d\eta\, R = D_0^2 \frac{\partial^4 P}{\partial u^4} - D_0 \frac{\partial^2}{\partial u^2} f \frac{\partial P}{\partial u} - D_0 \frac{\partial^2}{\partial u^2}\frac{\partial P}{\partial t}$$

For $\partial P/\partial t$ one may insert the lowest order expression given by the first two terms of (23). The third and fourth derivatives cancel out and the result is (23). The sole correction term of order τ_c has the form (16) with $n=2$, $m=1$, $k=0, 1$. Of course (23) agrees with (A7) to first order in τ_c.

APPENDIX C. CONFRONTATION WITH THE LITERATURE

Results will be quoted in our notation and adapted to the simplified version (9). It is of course easy to transform all equations back to the original x if so desired.

Sancho *et al.*[2] obtained (10), with

$$D(u) = D_0[1 + \tau_c f'(u)] \tag{C1}$$

This agrees with (14) to lowest order beyond Fokker–Planck. An estimate of the size of this correction is $D_0 \tau_c f'/\Delta^2 = (D_0/\Delta^2)\, \tau_c/\tau_m$. The fourth term of the primary expansion for non-Gaussian $\xi(t)$ can be estimated by $D_0^2 \tau_c/\Delta^4$. The relative size of both terms depends on the P considered. In equilibrium $\Delta^2 = D_0 \tau_m$ and both terms are equally important. It is therefore inconsistent to use (C1) for computing P^s without improving on the Fokker–Planck approximation—as mentioned in Section 9 and in ref. 24.

Fox[7] obtained (10) with

$$D(u) = D_0[1 - \tau_c f'(u)]^{-1} \tag{C2}$$

which also agrees with (14) to order τ_c. The higher orders do not agree, but they are meaningless anyway because of the neglected higher orders of the primary expansion.

Some authors[6,26] have simplified such equations by replacing the functions of u that occur in the coefficients with their averages. In this way (C2) becomes an as yet unknown function of t,

$$D(u) \to D_0[1 - \tau_c \langle f'(u) \rangle]^{-1}$$

This has the effect that the equation becomes nonlinear in P and that the local rate of change of P depends on the value of P everywhere else. It is clear that this cannot be correct for very flat P, let alone for a P consisting of two peaks. But even for a single peak this leads to incorrect higher orders.[27]

For the case that $\xi(t)$ is Ornstein–Uhlenbeck it was shown by Sancho *et al.*[2] that the function $D(u)$ obeys

$$[1 - \tau_c f'(u)]\, D(u) + \tau_c f(u)\, D'(u) = 1 \tag{C3}$$

This is easy to verify with the aid of the exact expression (14) and of course (4),

$$\begin{aligned}
D'(u) &= \alpha^2 \int_0^\infty e^{-\tau/\tau_c}\, d\tau \frac{d}{du}\frac{f(u)}{f(u^{-\tau})} \\
&= \alpha^2 f'(u) \int_0^\infty e^{-\tau/\tau_c} \frac{d\tau}{f(u^{-\tau})} - \alpha^2 f(u) \int_0^\infty e^{-\tau/\tau_c}\, d\tau \frac{f'(u^{-\tau})}{f(u^{-\tau})^2}\frac{du^{-\tau}}{du} \\
&= \frac{f'(u)}{f(u)} D(u) - \alpha^2 \int_0^\infty e^{-\tau/\tau_c}\, d\tau \frac{f'(u^{-\tau})}{f(u^{-\tau})} \\
&= \frac{f'(u)}{f(u)} D(u) + \alpha^2 \int_0^\infty e^{-\tau/\tau_c}\, d\tau \frac{1}{f(u^{-\tau})^2}\frac{d}{d\tau} f(u^{-\tau}) \\
&= \frac{f'(u)}{f(u)} D(u) - \alpha^2 e^{-\tau/\tau_c} \frac{1}{f(u^{-\tau})}\Bigg|_0^\infty - \frac{\alpha^2}{\tau_c}\int_0^\infty e^{-\tau/\tau_c} \frac{d\tau}{f(u^{-\tau})} \\
&= \frac{f'(u)}{f(u)} D(u) + \frac{\alpha^2}{f(u)} - \frac{1}{\tau_c f(u)} D(u)
\end{aligned}$$

This is the same as (C3). Note that ξ does not need to be Gaussian, it is sufficient that its autocorrelation function is exponential.

It is possible to write (C3) in the form

$$\frac{D(u)}{f(u)} + \tau_c f(u) \frac{d}{du}\frac{D(u)}{f(u)} = \frac{1}{f(u)} \tag{C4}$$

and to deduce from this

$$D(u) = f(u)\left[1 + \tau_c f(u)\frac{d}{du}\right]^{-1} \frac{1}{f(u)}$$

This form was proposed by Lindenberg and West.[4] Its expansion in powers of τ_c provides the correct series, namely (15) in which $\kappa(t)$ is taken from (4). It is true, of course, that (C4) determines $D(u)/f(u)$ only up to a solution of the homogeneous equation, but such a solution has a factor e^{-1/τ_c} and therefore does not contribute to the expansion.

APPENDIX D. AN EXAMPLE

As an example, consider the process u defined by

$$\dot{u} = -u + \xi(t), \qquad u(0) = u_0 \tag{D1}$$

where $\xi(t)$ has zero mean and autocorrelation function (4). The equation is linear and additive and can therefore be solved exactly in the way of Uhlenbeck and Ornstein. The result is, for the first two moments,

$$\langle u(t)\rangle = u_0 e^{-t} \tag{D2}$$

$$\langle u(t)^2\rangle = \langle u(t)\rangle^2 + \frac{\alpha^2\tau_c}{1+\tau_c} - \frac{\alpha^2\tau_c}{1-\tau_c} e^{-2t} + \frac{2\alpha^2\tau_c^2}{1-\tau_c^2} e^{-t-t/\tau_c} \tag{D3}$$

For later comparison note that these moments obey the equations

$$\partial_t\langle u(t)\rangle = -\langle u(t)\rangle \tag{D4}$$

$$\partial_t\langle u(t)^2\rangle = -2\langle u(t)^2\rangle + \frac{2\alpha^2\tau_c}{1+\tau_c}(1-e^{-t-t/\tau_c}) \tag{D5}$$

In the same way one finds the autocorrelation function

$$\langle u(t)\, u(t+\tau)\rangle = e^{-2t-\tau}\left(u_0^2 - \frac{\alpha^2\tau_c}{1-\tau_c}\right) + \frac{\alpha^2\tau_c}{1-\tau_c^2} e^{-\tau} - \frac{\alpha^2\tau_c^2}{1-\tau_c^2} e^{-\tau/\tau_c} + \frac{\alpha^2\tau_c}{1-\tau_c^2}(e^{-\tau} + e^{-\tau/\tau_c})\, e^{-t-t/\tau_c}$$

Here $\tau \geqslant 0$ and of course $t \geqslant 0$. It obeys the equation

$$\partial_t\langle u(t)\, u(t+\tau)\rangle = -\langle u(t)\, u(t+\tau)\rangle + \frac{\alpha^2\tau_c}{1+\tau_c}(1-e^{-t-t/\tau_c})\, e^{-\tau/\tau_c}$$

Now apply the cumulant expansion to this example. It leads to the Fokker–Planck equation (10) with coefficient (11):

$$\frac{\partial P(u,t)}{\partial t} = \frac{\partial}{\partial u} uP + \frac{\alpha^2}{1+1/\tau_c}\frac{\partial^2 P}{\partial u^2} \tag{D6}$$

From this one finds again (D4) and therefore also (D2). The second moment now obeys the equation

$$\partial_t\langle u(t)^2\rangle = -2\langle u(t)^2\rangle + \frac{2\alpha^2\tau_c}{1+\tau_c} \tag{D7}$$

It coincides with (D5) after an initial transient time of order τ_c. Also, after this transient time the exact second moment (D3) is a solution of (D7). Yet this is not what one would get by solving (D7) with the correct initial condition $\langle u(0)^2 \rangle = u_0^2$. For, the initial slip during the transient time creates an error in $\langle u(t)^2 \rangle$,

$$\frac{2\alpha^2\tau_c^2}{1-\tau_c^2}\left(e^{-2t} - e^{-t-t/\tau_c}\right)$$

which dies out only at the macroscopic relaxation rate of the system (D1).

This indicates already that after the transient time the Markov process $\tilde{u}(t)$ defined by (D6) does not coincide with the actual process u, not even as far as their second moments are concerned. Their autocorrelation functions remain different even at large t. The exact expression is, for $\tau > 0$,

$$\begin{aligned}\langle u(t)\, u(t+\tau) \rangle = {} & e^{-2t-\tau}\left(u_0^2 - \frac{\alpha^2\tau_c}{1-\tau_c}\right) + \frac{\alpha^2\tau_c}{1-\tau_c^2}\, e^{-\tau} \\ & - \frac{\alpha^2\tau_c^2}{1-\tau_c^2}\, e^{-\tau/\tau_c} + \frac{\alpha^2\tau_c^2}{1-\tau_c^2}\left(e^{-\tau} + e^{-\tau/\tau_c}\right) e^{-t-t/\tau_c}\end{aligned} \tag{D8}$$

which obeys with respect to τ the differential equation

$$\partial_\tau \langle u(t)\, u(t+\tau) \rangle = -\langle u(t)\, u(t+\tau) \rangle + \frac{\alpha^2\tau_c}{1+\tau_c}\left(1 - e^{-t-t/\tau_c}\right) e^{-\tau/\tau_c} \tag{D9}$$

On the other hand, the autocorrelation function of $\tilde{u}(t)$ obeys of course the same equation as $\langle \tilde{u} \rangle$,

$$\partial_\tau \langle \tilde{u}(t)\, \tilde{u}(t+\tau) \rangle = -\langle \tilde{u}(t)\, \tilde{u}(t+\tau) \rangle$$

The missing last term of (D9) does not disappear for large t, but there remains an error proportional to $e^{-\tau/\tau_c}$. One may also compare the explicit expression (D8) for the exact u with

$$\langle \tilde{u}(t)\, \tilde{u}(t+\tau) \rangle = e^{-2t-\tau}\left(u_0^2 - \frac{\alpha^2\tau_c}{1+\tau_c}\right) + \frac{\alpha^2\tau_c}{1+\tau_c}\, e^{-\tau}$$

for the Markov process defined by (D6). Even for large t there remains a difference proportional to $e^{-\tau}$.

Equation (D6) is the first term of the primary expansion. Its diffusion coefficient is

$$D(u) = \frac{\alpha^2\tau_c}{1+\tau_c} = \frac{D_0}{1+\tau_c} \tag{D10}$$

The secondary expansion consist in expanding this in powers of τ_c. The question of whether it is consistent to include the higher powers of τ_c while terminating the primary expansion can be investigated for the case that ξ is Ornstein–Uhlenbeck. In that case Eq. (A7) is valid to order τ_c^2; since in our linear example the last term of (A7) vanishes, it follows that (D6) is correct to order τ_c^2:

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial u}uP+D_0(1-\tau_c+\tau_c^2)\frac{\partial^2 P}{\partial u^2}+O(\tau_c^3)$$

But of course this is a very special case in that the equation is linear and the noise Ornstein–Uhlenbeck.

Consider the same equation (D1) but with a two-valued Markovian $\xi(t)$,

$$\xi(t)=\pm\alpha, \qquad \langle\xi(t)\rangle=0, \qquad \langle\xi(t_1)\,\xi(t_2)\rangle=\alpha^2 e^{-|t_1-t_2|/\tau_c}$$

It has the property that for an even set of successive times

$$\begin{aligned}&\langle\xi(t_1)\,\xi(t_2)\cdots\xi(t_{2p})\rangle\\&\quad=\xi(t_1)\,\xi(t_2)\rangle\langle\xi(t_3)\,\xi(t_4)\rangle\cdots\langle\xi(t_{2p-1})\,\xi(t_{2p})\rangle\end{aligned} \tag{D11}$$

The first two terms of the primary expansions are again given by (D6), because that equation only involved the autocorrelation function of $\xi(t)$. To determine the next term (A3) we write it again in the form (A4). In the present example $u^{-\tau}=ue^{\tau}$, so that on each line of (A4) the factors involving the derivatives reduce to

$$e^{-\tau_1-\tau_2-\tau_3}\nabla^4$$

Moreover, it follows from (D11) that the first line of (A4) vanishes. To evaluate the second line, one needs the integral

$$\int_0^\infty d\tau_1\int_{\tau_1}^\infty d\tau_2\int_{\tau_2}^\infty d\tau_3\,e^{-\tau_2/\tau_c}e^{-(\tau_3-\tau_1)/\tau_c}e^{-\tau_1-\tau_2-\tau_3}=\frac{\tau_c^3}{2(1+\tau_c)^2\,(1+3\tau_c)}$$

The integral on the third line gives the same. Consequently, we find [comp. (D6) and (D10)]

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial u}uP+\frac{D_0}{1+\tau_c}\frac{\partial^2 P}{\partial u^2}+\frac{D_0^2\tau_c}{(1+\tau_c)^2\,(1+3\tau_c)}\frac{\partial^4 P}{\partial u^4}$$

The new term is of type (16) with $n=4$, $m=0$, $k=0$.

The stationary solution is roughly $P^s\sim\exp(-u^2/2D_0)$, so that $\Delta=D_0^{1/2}$. Hence for this P^s the latter two terms are of order $(1+\tau_c)^{-1}$ and τ_c, respectively. Thus, for the present example it is *not* consistent to take the factor $(1+\tau_c)^{-1}$ in the second term seriously while neglecting the third term.

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